

Numerical Bayesian quantum-state assignment for a three-level quantum system

II. Average-value data with a constant, a Gaussian-like, and a Slater prior

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This paper offers examples of concrete numerical applications of Bayesian quantum-state assignment methods to a three-level quantum system. The statistical operator assigned on the evidence of various measurement data and kinds of prior knowledge is computed partly analytically, partly through numerical integration (in eight dimensions) on a computer. The measurement data consist in the average of outcome values of N identical von Neumann projective measurements performed on N identically prepared three-level systems. In particular the large- N limit will be considered. Three kinds of prior knowledge are used: one represented by a plausibility distribution constant in respect of the convex structure of the set of statistical operators; another one represented by a prior studied by Slater, which has been proposed as the natural measure on the set of statistical operators; the last prior is represented by a Gaussian-like distribution centred on a pure statistical operator, and thus reflecting a situation in which one has useful prior knowledge about the likely preparation of the system. The assigned statistical operators obtained with the first two kinds of priors are compared with the one obtained by Jaynes' maximum entropy method for the same measurement situation.

In the companion paper the case of measurement data consisting in absolute frequencies is considered.

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1. INTRODUCTION

In this paper we continue our two-part study [1] with examples of concrete numerical applications of Bayesian quantum-state assignment methods to a three-level quantum system. Since we will consider the same scenario as in the first paper, to avoid repeating ourselves we therefore refer the reader to the first paper for a more detailed and complete account of the motivations, explanations, discussions and references on the background, theory, formulas, nomenclature, etc, used in this paper. The main difference between the two papers lies in the type of measurement data considered. In the first paper the measurement data consisted in absolute frequencies of the outcomes of N identical von Neumann projective measurements performed on N identically prepared three-level systems. Here we will consider the same measurement situation, but the measurement data will instead be in the form of an average of values being associated to the measurement outcomes, in particular 1, 0, and -1 . The statistical operator encoding the average value data and prior knowledge is computed partly numerically and partly analytically in the limit when $N \rightarrow \infty$, for a constant, and also for two different kinds of a non-constant, prior probability distribution, and different average value data. A reason for studying data of this kind, other than the obvious one that it may have been given to us in this form, is that it constitutes an example of more complex data than mere absolute frequencies. It is also interesting to study this particular kind of data since it enables us to compare our assigned statistical operators with those obtained by instead using Jaynes' maximum entropy method [2] for the same measurement situation. The reason for doing this is that

we want to investigate whether or not this method could be seen as a special case of Bayesian quantum-state assignment, and if so, try to find the prior that would lead to the same statistical operator as the one obtained by using the maximum entropy method.

2. THE PRESENT STUDY

In this paper we study data D and prior knowledge I of the following kind:

- The measurement data D consist in the average of N outcome values of N instances of the same measurement performed on N identically prepared systems. The measurement is represented by the extreme positive-operator-valued measure (i.e., non-degenerate 'von Neumann measurement') having three possible distinct outcomes $\{1, 2, 3\}$ represented by the eigenprojectors $\{|1\rangle\langle 1|, |0\rangle\langle 0|, |-1\rangle\langle -1|\}$, where the eigenprojectors are labelled by their associated outcome values $\{1, 0, -1\}$, respectively. We consider the limiting case of very large N .
- Three different kinds of prior knowledge I are used. Two of them, I_{co} and I_{ga} , are the same as those given in first paper, i.e. a prior plausibility distribution

$$p(\rho | I_{\text{co}}) d\rho = g_{\text{co}}(\rho) d\rho \propto d\rho, \quad (1)$$

which is constant in respect of the convex structure of the set of statistical operators, in the sense explained in [1, § 3.4]; and a spherically symmetric, Gaussian-like prior distribution

$$p(\rho | I_{\text{ga}}) d\rho = g_{\text{ga}}(\rho) d\rho \propto \exp\left\{-\frac{\text{tr}[(\rho - \hat{\rho})^2]}{s^2}\right\} d\rho, \quad (2)$$

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centred on the statistical operator $\hat{\rho}$. The latter prior expresses some kind of knowledge that leads us to assign a higher plausibility to regions in the vicinity of $\hat{\rho}$. For this prior we consider two examples, when $\hat{\rho} = |1\rangle\langle 1|$ and $\hat{\rho} = |0\rangle\langle 0|$.¹

The third kind of prior knowledge, I_{sl} , is represented by the prior plausibility distribution

$$p(\rho|I_{\text{sl}}) d\rho = g_{\text{sl}}(\rho) d\rho \propto (\det \rho)^{2d+1} d\rho, \quad (3)$$

the so called ‘‘Slater prior’’ for a d -level system, which has been proposed as a candidate for being the appropriate measure on the set of statistical operators. [3]

The paper is organised as follows: In § 3 we present the reasoning leading to the statistical-operator-assignment formulae in the case of average value data, for finite N and in the limit when $N \rightarrow \infty$. We arrived at the same formulae (as special cases of formulae applicable to generic, not necessarily quantum-theoretical systems) in a series of papers [4, 5, 6]. In § 4 we present the particular case studied in this paper and give the statistical-operator-assignment formulae in this case, introduce the Bloch vector parametrisation, present the calculations by symmetry arguments and by numerical integration, discuss the result and in some cases compare it with that obtained by the maximum entropy method. Finally, the last section summarises and discusses the main points and results.

3. STATISTICAL OPERATOR ASSIGNMENT

3.1. General case

Again we assume there is a preparation scheme that produces quantum systems always in the same ‘condition’ — the same ‘state’ — where each condition is associated with a statistical operator. Suppose we come to know that N measurements, represented by the N positive-operator-valued measures $\{\mathbf{E}_\mu^{(k)} : \mu = 1, \dots, r_k\}$, $k = 1, \dots, N$, are or have been performed on N systems for which our knowledge I holds. In this paper we will be analysing the case when the data is an average of a number of outcome values and it will therefore be natural to limit ourselves to the situation when the N measurements are all instances of the same measurement. Thus, for all $k = 1, \dots, N$, $\{\mathbf{E}_\mu^{(k)}\} = \{\mathbf{E}_\mu\}$.

Let us say that the outcomes $i_1, \dots, i_k, \dots, i_N$ are or were obtained. Since every outcome is associated to an outcome value m_i , the average of all outcome values is

$$\bar{m} \equiv \sum_{k=1}^N m_{i_k} / N. \quad (4)$$

We will consider the general situation in which the data D consists in the knowledge that the average value \bar{m} in N repetitions of the measurement lies in a set \mathcal{Y} ;

$$D \triangleq [\bar{m} \in \mathcal{Y}]. \quad (5)$$

Such kind of data arise when the measurements is affected by uncertainties and is moreover ‘‘coarse-grained’’ for practical purposes, so that not precise average values are obtained but rather a region of possible ones. On the evidence of D we can update the prior plausibility distribution $g(\rho) d\rho := p(\rho|I) d\rho$. By the rules of plausibility theory²

$$p(\rho|D \wedge I) d\rho = \frac{p(D|\rho) g(\rho) d\rho}{\int_{\mathbb{S}} p(D|\rho) g(\rho) d\rho}, \quad (6)$$

where \mathbb{S} is the set of all statistical operators.

The plausibility of obtaining a particular sequence of outcomes is

$$p(\mathbf{E}_{i_1}, \dots, \mathbf{E}_{i_N} | \rho) = \prod_{i=1}^r [\text{tr}\{\mathbf{E}_i \rho\}]^{N_i}, \quad (7)$$

with the convention, here and in the following, that only factors with $N_i > 0$ are to be multiplied over, and where we have used that $\text{tr}\{\mathbf{E}_i \rho\} = p(\mathbf{E}_i | \rho)$ and $(N_1, \dots, N_r) =: \bar{N}$ are the absolute frequencies of appearance of the r possible outcomes (naturally, $N_i \geq 0$ and $\sum_i N_i = N$). Since the exact order of the sequence of outcomes is unimportant and only the absolute frequencies of appearance \bar{N} matter, the plausibility of the absolute frequencies \bar{N} in N measurements is

$$p(\bar{N} | \rho) = N! \prod_{i=1}^r \frac{[\text{tr}\{\mathbf{E}_i \rho\}]^{N_i}}{N_i!}. \quad (8)$$

Define \mathbb{N}_N^r as the set of all absolute frequencies \bar{N} , for fixed N and r . By the rules of plausibility theory we then have that

$$p(D | \rho) = \sum_{\bar{N} \in \mathbb{N}_N^r} p(D | \bar{N} \wedge \rho) p(\bar{N} | \rho). \quad (9)$$

Given that we know \bar{N} , we can with certainty tell if \bar{N} corresponds to an average value

$$\bar{m} \equiv \sum_{i=1}^r N_i m_i / N \quad (10)$$

that belongs to the set \mathcal{Y} , and knowledge of the statistical operator ρ is here irrelevant. We thus have that $p(D | \bar{N} \wedge \rho) = p(\bar{m} \in \mathcal{Y} | \bar{N}) = 1$ if $\bar{N} \in \phi_{\mathcal{Y}}$ and 0 otherwise, where we have defined

$$\phi_{\mathcal{Y}} := \{\bar{N} \in \mathbb{N}_N^r | \sum_i N_i m_i / N \in \mathcal{Y}\}. \quad (11)$$

¹ Note that the case $\hat{\rho} = |-1\rangle\langle -1|$ is equivalent to the case with $\hat{\rho} = |1\rangle\langle 1|$.

² We do not explicitly write the prior knowledge I whenever the statistical operator appears on the conditional side of the plausibility; i.e., $p(\cdot | \rho) := p(\cdot | \rho, I)$.

Using this together with equations (8) and (9) we obtain:

$$p(D|\rho) = \sum_{\tilde{N} \in \phi_\gamma} p(\tilde{N}|\rho) = N! \sum_{\tilde{N} \in \phi_\gamma} \prod_{i=1}^r \frac{[\text{tr}\{E_i \rho\}]^{N_i}}{N_i!}. \quad (12)$$

Inserting this into equation (6) we finally obtain:

$$p(\rho|D \wedge I) d\rho = \frac{\sum_{\tilde{N} \in \phi_\gamma} \left(\prod_{i=1}^r \frac{[\text{tr}\{E_i \rho\}]^{N_i}}{N_i!} \right) g(\rho) d\rho}{\sum_{\tilde{N} \in \phi_\gamma} \int_{\mathbb{S}} \left(\prod_{i=1}^r \frac{[\text{tr}\{E_i \rho\}]^{N_i}}{N_i!} \right) g(\rho) d\rho}. \quad (13)$$

We saw in the first paper that generic knowledge \tilde{I} can be represented by or “encoded in” a unique statistical operator:

$$\rho_{\tilde{I}} := \int_{\mathbb{S}} \rho p(\rho|\tilde{I}) d\rho. \quad (14)$$

The statistical operator encoding the joint knowledge $D \wedge I$ is thus given by

$$\rho_{D \wedge I} = \frac{\sum_{\tilde{N} \in \phi_\gamma} \int_{\mathbb{S}} \rho \left(\prod_{i=1}^r \frac{[\text{tr}\{E_i \rho\}]^{N_i}}{N_i!} \right) g(\rho) d\rho}{\sum_{\tilde{N} \in \phi_\gamma} \int_{\mathbb{S}} \left(\prod_{i=1}^r \frac{[\text{tr}\{E_i \rho\}]^{N_i}}{N_i!} \right) g(\rho) d\rho}. \quad (15)$$

3.2. Large- N limit

Let us now summarise some results obtained in [6] for the case of very large N . Consider the general situation in which each data set D_N consists in the knowledge that the relative frequencies $f \equiv (f_i) := (N_i/N)$ lie in a region $\Phi_N = \{f | [\sum_i f_i m_i] \in \gamma_N\}$, where γ_N is a region in which the average values lie (being such that Φ_N has a non-empty interior and its boundary has measure zero in respect of the prior plausibility measure). Mathematically we want to see what form the state-assignment formulae take in the limit $N \rightarrow \infty$. Consider a sequence of data sets $\{D_N\}_{N=1}^\infty$ with corresponding sequences of regions $\{\gamma_N\}_{N=1}^\infty$ and $\{\Phi_N\}_{N=1}^\infty$, and assume the regions converges (in a topological sense specified in [6]) to regions γ_∞ and Φ_∞ (the latter also with non-empty interior and with boundary of measure zero), respectively.

Given that the statistical operator is ρ , the plausibility distribution for the outcomes is

$$q(\rho) \equiv (q_i(\rho)) \quad \text{with} \quad q_i(\rho) := \text{tr}(E_i \rho). \quad (16)$$

In [6] it is shown that

$$p(\rho|D_N \wedge I) d\rho \propto \begin{cases} 0, & \text{if } q(\rho) \notin \Phi_\infty, \\ p(\rho|I) d\rho, & \text{if } q(\rho) \in \Phi_\infty, \end{cases} \quad \text{as } N \rightarrow \infty. \quad (17)$$

Further it is also shown that if γ_∞ degenerates into a single average value \bar{m} , the expression above becomes³

$$p(\rho|\bar{m} \wedge I) d\rho \propto p(\rho|I) \delta[\sum_i q_i(\rho) m_i - \bar{m}] d\rho. \quad (18)$$

This is an intuitively satisfying result, since in the limit when $N \rightarrow \infty$ we would expect that it is only those statistical operators ρ whose expectation value $\sum_i q_i(\rho) m_i$ is equal to the measured average value \bar{m} that could have been the case. The data single out a set of statistical operators, and these are then given weight according to the prior $p(\rho|I) d\rho = g(\rho) d\rho$, specified by us.

By normalising the posterior plausibility distribution in equation (18), the assigned statistical operator in equation (14) is then given by

$$\rho_{D \wedge I} = \frac{\int_{\mathbb{S}} \rho g(\rho) \delta[\sum_i q_i(\rho) m_i - \bar{m}] d\rho}{\int_{\mathbb{S}} g(\rho) \delta[\sum_i q_i(\rho) m_i - \bar{m}] d\rho}. \quad (19)$$

4. AN EXAMPLE OF STATE ASSIGNMENT FOR A THREE-LEVEL SYSTEM

4.1. Three-level case

We will now consider the particular case studied in this paper. The preparation scheme concerns three-level quantum systems; the corresponding set of statistical operators will be denoted by \mathbb{S}_3 . We are going to consider the case when the number of measurements N is very large and in the limit goes to infinity. The N measurements are all instances of the same measurement, namely a non-degenerate projection-valued measurement (often called ‘von Neumann measurement’). Thus, for all $k = 1, \dots, N$, $\{E_\mu^{(k)}\} = \{E_\mu\} := \{|1\rangle\langle 1|, |0\rangle\langle 0|, |-1\rangle\langle -1|\}$, where the projectors, labelled by the particular outcome values $(m_1, m_2, m_3) = (1, 0, -1)$ we have chosen to consider here, define an orthonormal basis in Hilbert space. All relevant operators will, quite naturally and advantageously, be expressed in this basis. We have for example that $q_\mu(\rho) = \text{tr}(E_\mu \rho) = \rho_{\mu\mu}$, the μ th diagonal element of ρ in the chosen basis. As data we are given that the average of the measurement outcome values is \bar{m} (more precisely in the sense that Φ_∞ degenerates into a single average value \bar{m}).

4.2. Bloch vector parametrisation and symmetries

We will be using the same parametrisation of the statistical operators as in the companion paper, i.e. in terms of Bloch

³ Note that we have here, with abuse of notation, written $p(\rho|\bar{m} \wedge I)$ instead of the more correct form $p(\rho|(\bar{m} = \bar{m}^*) \wedge I)$, to avoid introducing another variable \bar{m}^* for the average value data.

vectors \mathbf{x} . For a three-level system the Bloch vector expansion of a statistical operator $\rho(\mathbf{x})$ is given by:

$$\rho(\mathbf{x}) = \frac{1}{3}I_3 + \frac{1}{2}\sum_{j=1}^8 x_j \lambda_j, \quad (20)$$

where

$$x_i = \text{tr}\{\lambda_i \rho(\mathbf{x})\} \equiv \langle \lambda_i \rangle_{\rho(\mathbf{x})}. \quad (21)$$

The Gell-Mann operators λ_i are Hermitian and can therefore be regarded as observables. Note that our von Neumann measurement corresponds to the observable

$$\lambda_3 \equiv |1\rangle\langle 1| + 0|0\rangle\langle 0| - |-1\rangle\langle -1|. \quad (22)$$

Hence, given a statistical operator $\rho(\mathbf{x})$, the following holds for the expectation value of the outcome values $\{1, 0, -1\}$ for this particular measurement:

$$\langle \lambda_i \rangle_{\rho(\mathbf{x})} = \sum_i q_i(\mathbf{x}) m_i = \rho_{11}(\mathbf{x}) - \rho_{33}(\mathbf{x}) = x_3. \quad (23)$$

Equation (18) thus becomes

$$p[\mathbf{x} | \bar{m} \wedge I] d\mathbf{x} \propto g(\mathbf{x}) \delta(x_3 - \bar{m}) d\mathbf{x}, \quad (24)$$

and the assigned statistical operator in equation (19) assumes the form

$$\rho_{\bar{m} \wedge I} = \frac{\int_{\mathbb{B}_8} \rho(\mathbf{x}) g(\mathbf{x}) \delta(x_3 - \bar{m}) d\mathbf{x}}{\int_{\mathbb{B}_8} g(\mathbf{x}) \delta(x_3 - \bar{m}) d\mathbf{x}}, \quad (25)$$

where \mathbb{B}_8 is the set of all three-level Bloch vectors. This can be rewritten in a form especially suited for numerical integration by computer, which we shall use hereafter:

$$\rho_{\bar{m} \wedge I} = \frac{\int_{\mathbb{C}_8} \rho(\mathbf{x}) g(\mathbf{x}) \delta(x_3 - \bar{m}) \chi_{\mathbb{B}_8}(\mathbf{x}) d\mathbf{x}}{\int_{\mathbb{C}_8} g(\mathbf{x}) \delta(x_3 - \bar{m}) \chi_{\mathbb{B}_8}(\mathbf{x}) d\mathbf{x}}, \quad (26)$$

where $\chi_{\mathbb{B}_8}(\mathbf{x})$ is the characteristic function of the set $\mathbb{B}_8 \subset \mathbb{C}_8 := [-1, 1]^7 \times \left[-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$. Using the Bloch vector expansion in equation (20) we see that by computing the following set of integrals we have determined $\rho_{\bar{m} \wedge I}$:

$$L_i[\bar{m}, I] := \int_{\mathbb{C}_8} x_i g(\mathbf{x}) \delta(x_3 - \bar{m}) \chi_{\mathbb{B}_8}(\mathbf{x}) d\mathbf{x}, \quad (27)$$

where $i \in \{1, \dots, 8\}$, and

$$Z[\bar{m}, I] := \int_{\mathbb{C}_8} g(\mathbf{x}) \delta(x_3 - \bar{m}) \chi_{\mathbb{B}_8}(\mathbf{x}) d\mathbf{x}, \quad (28)$$

where the dependence of the average value and prior knowledge is indicated within brackets. The assigned statistical operator will then given by

$$\rho_{\bar{m} \wedge I} = \frac{1}{3}I_3 + \frac{1}{2} \sum_{i=1}^8 \frac{L_i[\bar{m}, I]}{Z[\bar{m}, I]} \lambda_i. \quad (29)$$

One sees directly from equations (27) and (28) that $L_3[\bar{m}, I]/Z[\bar{m}, I] = \bar{m}$ ($Z[\bar{m}, I]$ can never vanish, its integrand being positive and never identically naught).

For the same reasons already accounted for in the first paper we will not try to determine $\rho_{\bar{m} \wedge I}$ exactly, but also here compute it with a combination of symmetry considerations of \mathbb{B}_8 and numerical integration. For all three kinds of prior knowledge considered in this paper the same symmetry arguments used in the companion paper also holds here, so again we have that $L_i[\bar{m}, I]/Z[\bar{m}, I] = 0$ for all $i \neq 3, 8$ and any average value $-1 \leq \bar{m} \leq 1$. The assigned Bloch vector is thus given by $(0, 0, \bar{m}, 0, 0, 0, 0, L_8[\bar{m}, I]/Z[\bar{m}, I])$. This means that $\rho_{\bar{m} \wedge I}$ lies in the (x_3, x_8) -plane and it has, in the chosen eigenbasis, the diagonal matrix form

$$\rho_{\bar{m} \wedge I} = \begin{pmatrix} \frac{1}{3} + \frac{\bar{m}}{2} + \frac{L_8[\bar{m}, I]}{2\sqrt{3}Z[\bar{m}, I]} & 0 & 0 \\ 0 & \frac{1}{3} - \frac{L_8[\bar{m}, I]}{\sqrt{3}Z[\bar{m}, I]} & 0 \\ 0 & 0 & \frac{1}{3} - \frac{\bar{m}}{2} + \frac{L_8[\bar{m}, I]}{2\sqrt{3}Z[\bar{m}, I]} \end{pmatrix}. \quad (30)$$

4.3. Numerical integration, results and the maximum entropy method

We have used numerical integration⁴ to compute $L_8[\bar{m}, I]/Z[\bar{m}, I]$ for different prior knowledge and different values of \bar{m} . The result for a constant prior density is shown in figure 1, where the blue curve (with bars indicating the numerical-integration uncertainties) is the Bloch vector corresponding to $\rho_{\bar{m} \wedge I_{\text{co}}}$ plotted for different values of $x_3 = \bar{m}$.⁵

It is interesting to compare $\rho_{\bar{m} \wedge I_{\text{co}}}$ with the statistical operator obtained by the maximum entropy method [2] for the measurement situation we are considering here. Given the expectation value $\langle \mathbf{M} \rangle$ of a Hermitian operator \mathbf{M} , corresponding to an observable M , the maximum entropy method assigns the

⁴ Using quasi Monte Carlo-integration in Mathematica 5.2 on a PC (Pentium 4 processor, 3 GHz). The computation times are given in figures 1 to 4, and for more details on the numerical integration we again refer the reader to the companion paper [1].

⁵ Note that we have for all three kinds of priors considered in this paper computed $L_8[\bar{m}, I]/Z[\bar{m}, I]$ only for non-negative values of $x_3 = \bar{m}$, since by using the symmetry operation $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_6, x_7, -x_3, x_4, -x_5, x_1, x_2, x_8)$ one can show that $L_8[\bar{m}, I]/Z[\bar{m}, I]$ is invariant under a sign change of \bar{m} . Further, we have not computed $L_8[\bar{m}, I]/Z[\bar{m}, I]$ for $\bar{m} = \pm 1$, since it follows from [1, eq. 17] that $L_8[\bar{m}, I]/Z[\bar{m}, I] = 1/\sqrt{3}$ is the only possibility in this case (which one also realises by looking at the figures).

statistical operator to the system that maximises the von Neumann entropy $S := -\text{tr}\{\rho \ln \rho\}$ and satisfies the constraint $\text{tr}\{\rho \mathbf{M}\} = \langle \mathbf{M} \rangle$. Having obtained an average value \bar{M} from many instances of the same measurement performed on identically prepared systems, one conventionally sets $\langle \mathbf{M} \rangle = \bar{M}$.

In our case the operator \mathbf{M} would be identified as the Hermitian operator λ_3 and \bar{M} as \bar{m} . Hence the maximum entropy method corresponds here to an assignment of the statistical operator that maximises S among all statistical operators satisfying $\langle \lambda_3 \rangle = \bar{m}$, and this statistical operator is given by

$$\rho_{ME} := \frac{e^{-\mu(\bar{m}) \lambda_3}}{\text{tr}\{e^{-\mu(\bar{m}) \lambda_3}\}}, \quad (31)$$

where

$$\mu(\bar{m}) := \ln \left\{ \frac{-\bar{m} + \sqrt{4 - 3\bar{m}^2}}{2(\bar{m} + 1)} \right\}. \quad (32)$$

This could be compared with the statistical operator $\rho_{\bar{m} \wedge I}$ obtained by instead using Bayesian quantum-state assignment, and expressed in general form as in equation (25) it is seen to instead be given by a weighted sum, with weight $g(\mathbf{x}) d\mathbf{x}$, of all statistical operators with $\langle \lambda_3 \rangle = x_3 = \bar{m}$.

In the case of a constant prior one sees from figure 1 that $\rho_{\bar{m} \wedge I_{\text{co}}}$ is in general different from ρ_{ME} (the red curve [without bars]). This means for instance that, if the maximum entropy method is a special case of Bayesian quantum-state assignment, the statistical operator obtained by the former method corresponds to a non-constant prior probability distribution $g(\mathbf{x}) d\mathbf{x}$ on \mathbb{B}_8 in the latter method. This conclusion in itself is perhaps not so surprising, but it raises an interesting question: Does there exist a (non-constant on \mathbb{B}_8) prior distribution $g(\rho) d\rho$ that one with Bayesian quantum-state assignment in general obtains the same assigned statistical operator as with the maximum entropy method?

A strong candidate is the ‘‘Bures prior’’ which has been proposed as the natural measure on the set of all statistical operators (see e.g. [7, 8, 9, 10, 11]), but unfortunately it turns out to be difficult to do numerical integrations on it due to its complicated functional form, so we have not computed the assigned statistical operator in this case. Another interesting candidate is the ‘‘Slater prior’’ [3], which have also been suggested to be the natural measure on the set of all statistical operators, and the computed assigned statistical operator in this case is shown in figure 2. One can see directly from the figure that although it is similar to the curve obtained by the maximum entropy method, we have found them to differ.

The computed assigned statistical operators for the Gaussian-like prior, centred on the projectors $\hat{\rho} = |1\rangle\langle 1|$ and $\hat{\rho} = |0\rangle\langle 0|$ with ‘‘breadth’’ $s = 1/4$, are shown in figures 3 and 4, respectively. Apart from being symmetric under a sign change of \bar{m} , as already have been noted in footnote 5, one can also show that $L_8[\bar{m}, I_{\text{ga}}]/Z[\bar{m}, I_{\text{ga}}]$ does not depend on the \hat{x}_3 -coordinate of the statistical operator the prior is centred on.

5. CONCLUSIONS

This was the second paper in a two-part study where the Bayesian quantum-state assignment methods has been applied to a three-level system, showing that the numerical implementation is possible and simple in principle. This paper should not only be of theoretical interest but also be of use to experimentalists involved in state estimation. We have analysed the situation where we are given the average of outcome values from N repetitions of identical von Neumann projective measurements performed on N identically prepared three-level systems, when the number of repetitions N becomes very large. From this measurement data together with different kinds of prior knowledge of the preparation, a statistical operator can be assigned to the system. By a combination of symmetry arguments and numerical integration we computed the assigned statistical operator for different average values and for a constant, and also for two examples of a non-constant, prior probability distribution.

The results were also compared with that obtained by the maximum entropy method. An interesting question is whether there exists a prior probability distribution that gives rise to an assigned statistical operator which is in general identical to the one given by the maximum entropy method, i.e. if the maximum entropy method could be seen as a special case of Bayesian quantum-state assignment? In the case of a constant and a ‘‘Slater prior’’ on the Bloch vector space of a three-level system we saw that the assigned statistical operator did not agree with the one given by the maximum entropy method. It would therefore be interesting to try other kinds of priors, in particular ‘‘special’’ priors like the Bures one.

The generalisation of the present study to data involving different kinds of measurement is straightforward. Of course, in the general case one has to numerically determine a greater number of parameters (the $L_j[\bar{m}, I]$) and therefore compute a greater number of integrals.

Post scriptum: During the preparation of this manuscript, P. Slater kindly informed us that some of the integrals numerically computed here and in the previous paper can in fact be calculated analytically, using *cylindrical algebraic decomposition* [12, 13, 14, 15, 16] with a parametrisation introduced by Bloore [17]; cf. Slater [18]. This is true, e.g., for the integrals involving the constant and Slater’s priors. By this method Slater has also proven the exact validity of eq. (52) of our previous paper [1]. We plan to use and discuss more extensively this method in later versions of these papers.

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(Note: ‘arxiv eprints’ are located at <http://arxiv.org/>.)

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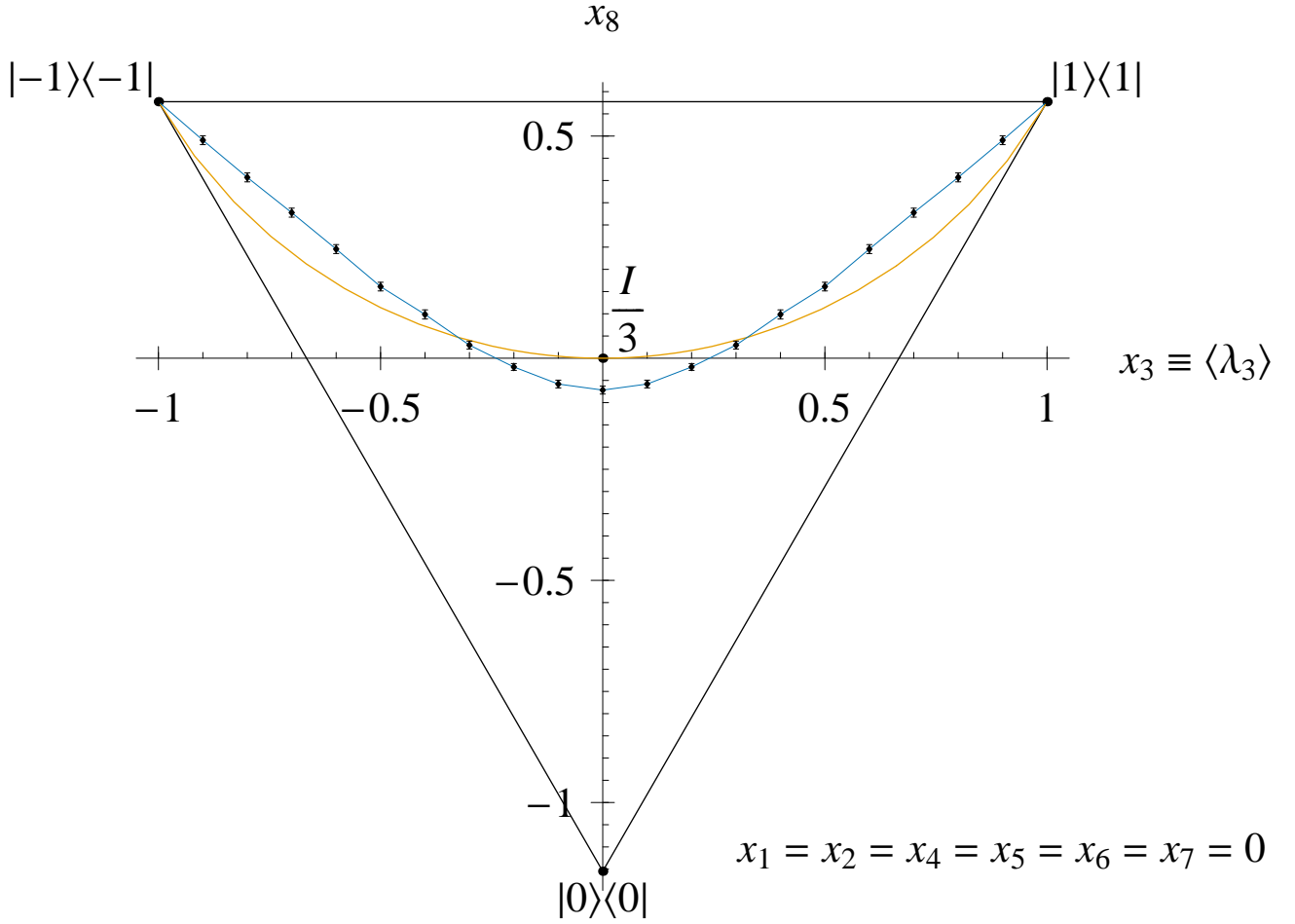


Figure 1: Bloch vectors of the assigned statistical operator for prior knowledge I_{co} , computed by numerical integration for different average values $\bar{m} \equiv x_3 \equiv \langle \lambda_3 \rangle$ (connected by the blue curve [with bars]). The red curve (without bars) is the statistical operator given by the maximum entropy method also as a function of the average value $\bar{m} \equiv x_3 \equiv \langle \lambda_3 \rangle$. The large triangle is the two-dimensional cross section of the set \mathbb{B}_8 along the plane Ox_3x_8 . The maximum numerical-integration uncertainty in the x_8 component is ± 0.01 . Note that only the ten points for $0 \leq \bar{m} \leq 0.9$ have been determined by numerical integration, since the nine points for $-0.9 \leq \bar{m} \leq -0.1$ can be exactly determined from the former by symmetry arguments. The endpoints corresponding to $\bar{m} = \pm 1$ were set manually, since $x_8 = 1/\sqrt{3}$ is the only possibility in this case. Within the given uncertainties, numerical computations yielded the exact results. The computation was done on a PC (Pentium 4 processor, 3 GHz) and the computation time was 15 min.

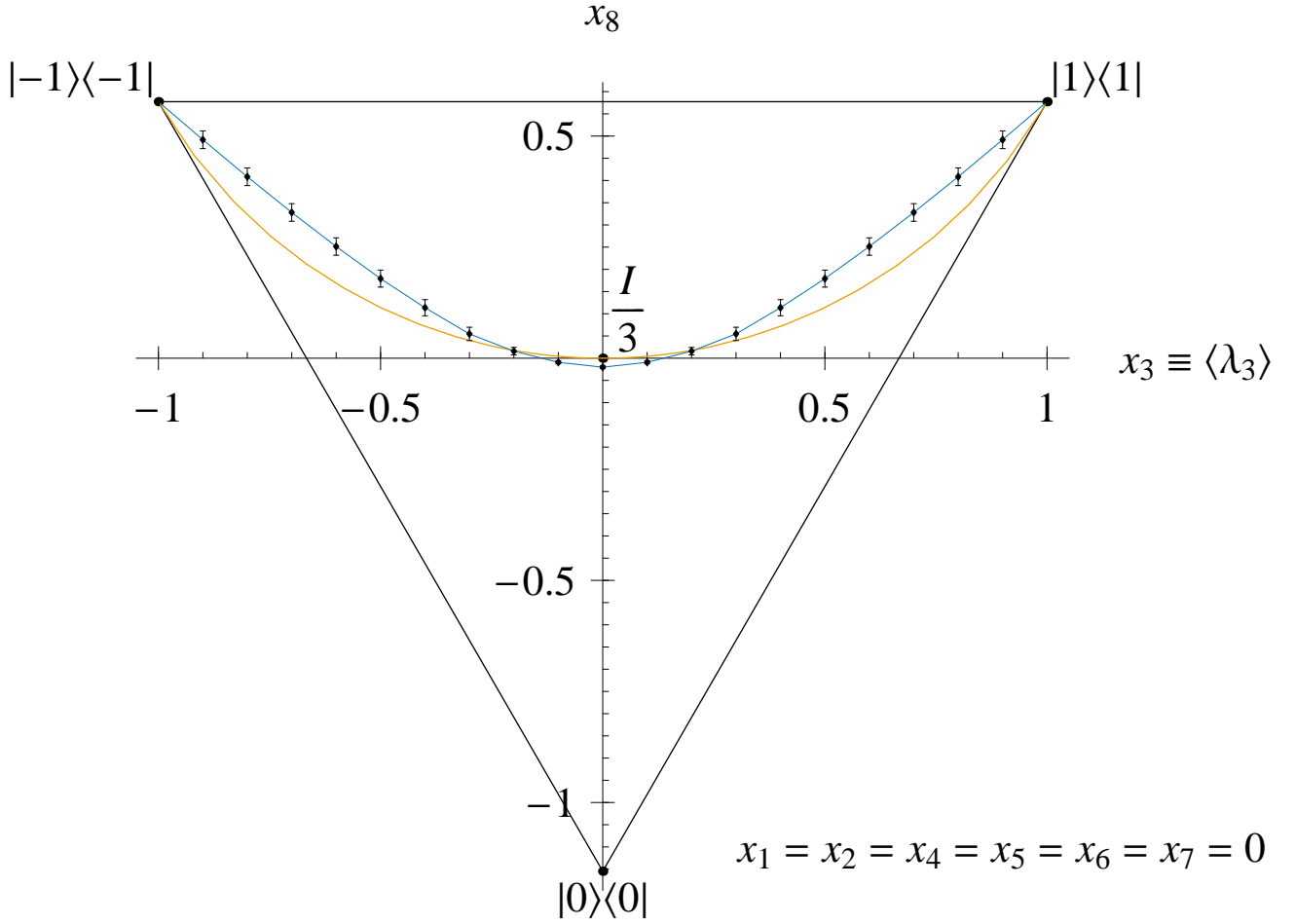


Figure 2: Bloch vectors of the assigned statistical operator for prior knowledge I_{sl} , computed by numerical integration for different average values $\bar{m} \equiv x_3 \equiv \langle \lambda_3 \rangle$ (connected by the blue curve [with bars]). The red curve (without bars) is the statistical operator given by the maximum entropy method also as a function of the average value $\bar{m} \equiv x_3 \equiv \langle \lambda_3 \rangle$. The large triangle is the two-dimensional cross section of the set \mathbb{B}_8 along the plane Ox_3x_8 . The maximum numerical-integration uncertainty in the x_8 component is ± 0.02 . Note that only the ten points for $0 \leq \bar{m} \leq 0.9$ have been determined by numerical integration, since the nine points for $-0.9 \leq \bar{m} \leq -0.1$ can be exactly determined from the former by symmetry arguments. The endpoints corresponding to $\bar{m} = \pm 1$ were set manually, since $x_8 = 1/\sqrt{3}$ is the only possibility in this case. Within the given uncertainties, numerical computations yielded the exact results. The computation was done on a PC (Pentium 4 processor, 3 GHz) and the computation time was 250 min.

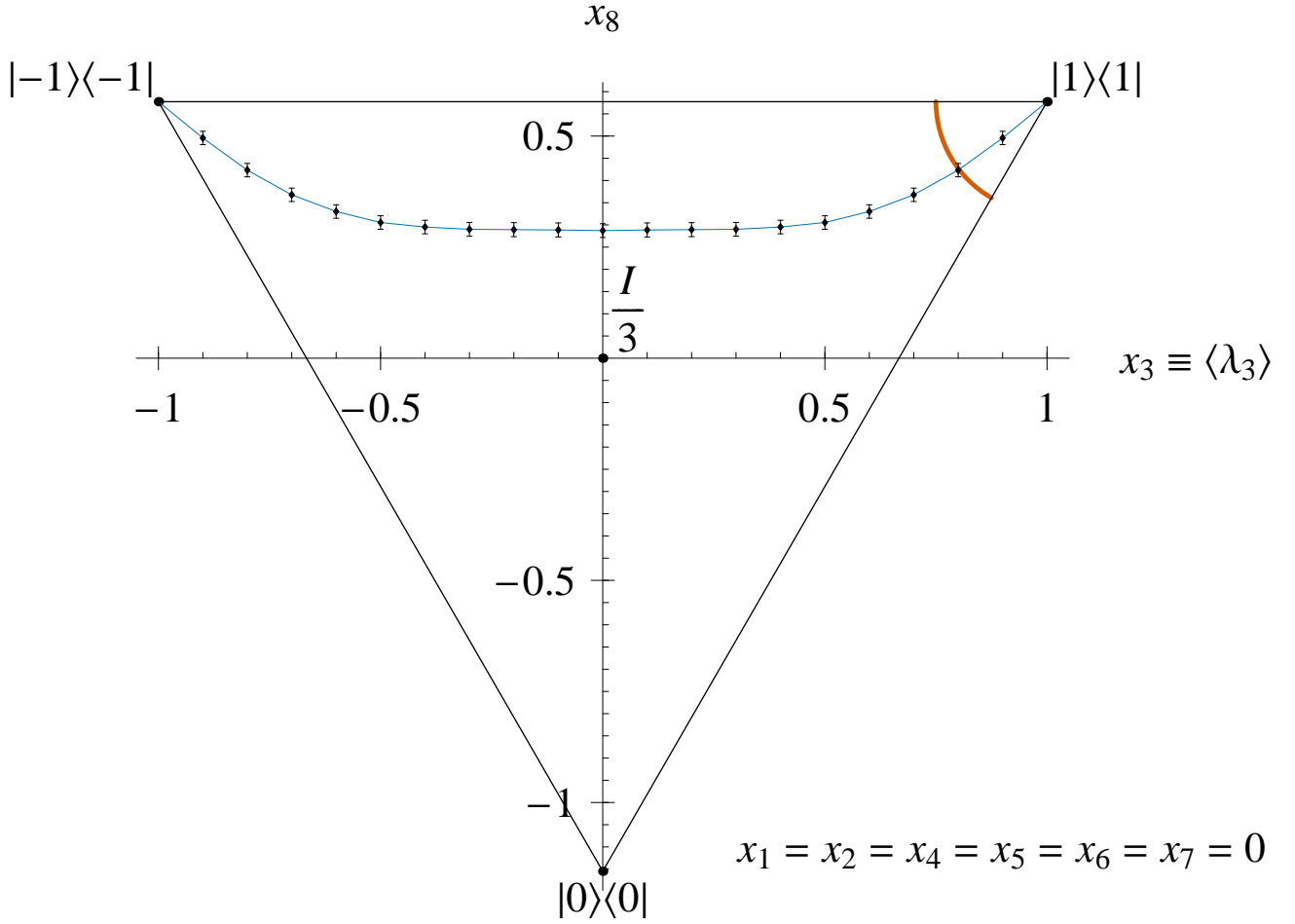


Figure 3: Bloch vectors of the assigned statistical operator for prior knowledge I_{ga} , computed by numerical integration for different average values $\bar{m} \equiv x_3 \equiv \langle \lambda_3 \rangle$ (connected by the curve). The large triangle is the two-dimensional cross section of the set \mathbb{B}_8 along the plane $\mathcal{O}_{x_3 x_8}$. The prior knowledge is represented by a Gaussian-like distribution of “breadth” $s = 1/4$ centred on the pure statistical operator $|1\rangle\langle 1|$. The small circular arc is the locus of the Bloch vectors (on the plane) at a distance $|\mathbf{x} - \hat{\mathbf{x}}| = s$ from the vector $\hat{\mathbf{x}} := (0, 0, 1, 0, 0, 0, 0, 1/\sqrt{3})$ corresponding to the statistical operator $|1\rangle\langle 1|$. The numerical-integration uncertainty in the x_8 component is ± 0.016 . Note that only the ten points for $0 \leq \bar{m} \leq 0.9$ have been determined by numerical integration, since the nine points for $-0.9 \leq \bar{m} \leq -0.1$ can be exactly determined from the former by symmetry arguments. The endpoints corresponding to $\bar{m} = \pm 1$ were set manually, since $x_8 = 1/\sqrt{3}$ is the only possibility in this case. Within the given uncertainties, numerical computations yielded the exact results. The computation was done on a PC (Pentium 4 processor, 3 GHz) and the computation time was 30 min.

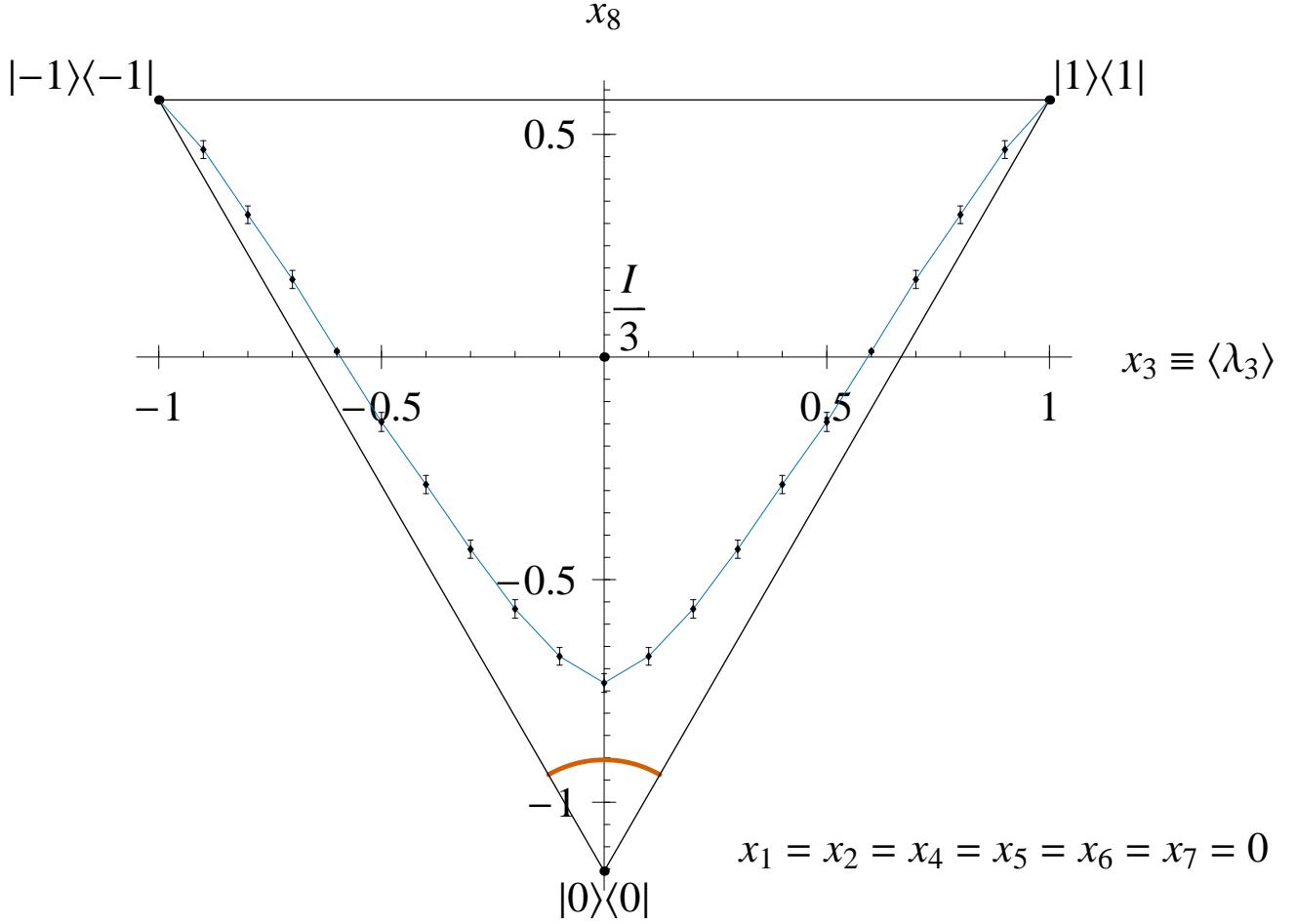


Figure 4: Bloch vectors of the assigned statistical operator for prior knowledge I_{ga} , computed by numerical integration for different average values $\bar{m} \equiv x_3 \equiv \langle \lambda_3 \rangle$ (connected by the curve). The large triangle is the two-dimensional cross section of the set \mathbb{B}_8 along the plane $\mathcal{O}_{x_3 x_8}$. The prior knowledge is represented by a Gaussian-like distribution of “breadth” $s = 1/4$ centred on the pure statistical operator $|0\rangle\langle 0|$. The small circular arc is the locus of the Bloch vectors (on the plane) at a distance $|\mathbf{x} - \hat{\mathbf{x}}| = s$ from the vector $\hat{\mathbf{x}} := (0, 0, 0, 0, 0, 0, 0, -2/\sqrt{3})$ corresponding to the statistical operator $|0\rangle\langle 0|$. The numerical-integration uncertainty in the x_8 component is ± 0.02 . Note that only the ten points for $0 \leq \bar{m} \leq 0.9$ have been determined by numerical integration, since the nine points for $-0.9 \leq \bar{m} \leq -0.1$ can be exactly determined from the former by symmetry arguments. The endpoints corresponding to $\bar{m} = \pm 1$ were set manually, since $x_8 = 1/\sqrt{3}$ is the only possibility in this case. Within the given uncertainties, numerical computations yielded the exact results. The computation was done on a PC (Pentium 4 processor, 3 GHz) and the computation time was 425 min.